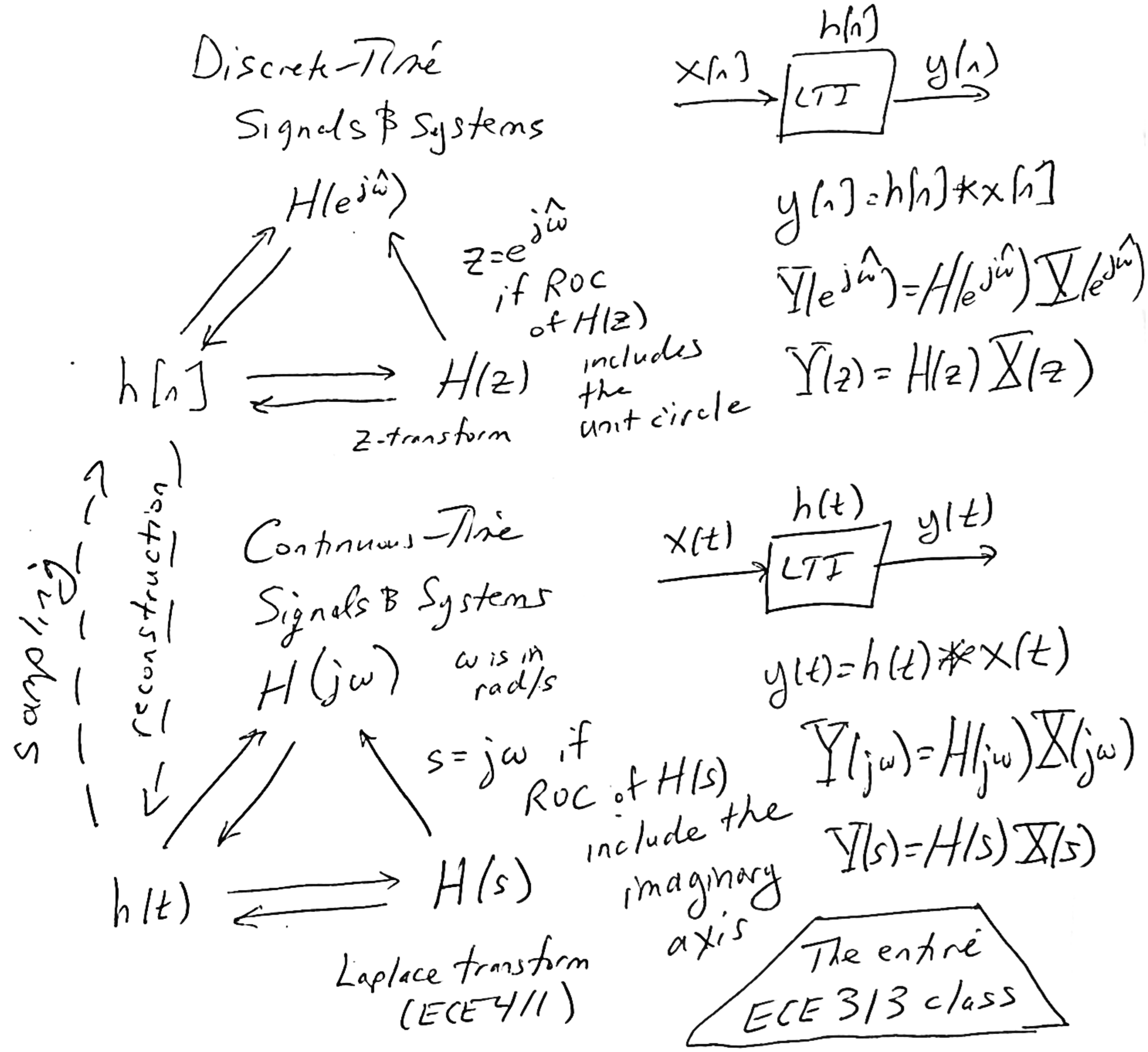
[11:00-] ECE 313 course content summarized in one page



[11:10] Discrete, continuous Fourier transform, z transform, Laplace transform

Discrete-Time Signal $x[n]$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\hat{\omega} n}$$

Fourier Transform

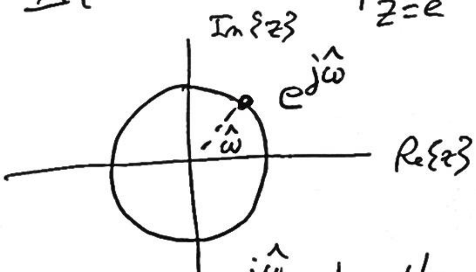
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

z-transform

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]\, (e^{j\hat{\omega}})^{-n}$$

When it's valid to do so,

$$X(e^{j\hat{\omega}}) = X(z)\Big|_{z=e^{j\hat{\omega}}}$$



Substitute $z=e^{j\hat{\omega}}$ when the region of convergence of the z-transform includes the unit circle.

Continuous-Time Signal $x(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Fourier Transform

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

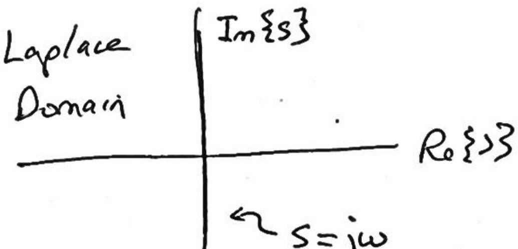
Laplace Transform

s is a complex variable.

When it's valid to so,

$$X(j\omega) = X(s)\Big|_{s=j\omega}$$

Laplace Domain



Substitute $s=j\omega$ when the region of convergence of the Laplace transform includes the imaginary axis.

Discrete-time	**Continuous-time**
<u>Fourier transform</u> $X\left(e^{j\hat{\omega}}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}$	<u>Fourier transform</u> $X(j\omega) = \int_{t=-\infty}^{\infty} x(t)e^{-j\omega t}dt$
<u>Z transform</u> $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$	<u>Laplace transform</u> $X(s) = \int_{t=-\infty}^{\infty} x(t)e^{-st}dt$
When it's valid to do so, $X\left(e^{j\hat{\omega}}\right) = X(z)\vert_{z=e^{j\hat{\omega}}}$	When it's valid to do so, $X(j\omega) = X(s)\vert_{s=j\omega}$

[11:20] Differences between continuous and discrete time

- Time variable: integer valued (discrete-time) or real-valued (continuous-time)
- Origin is included is included in discrete time
- Discrete-time frequency is 2π periodic

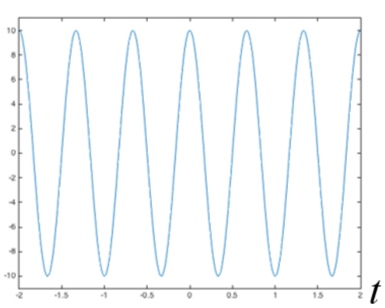
[11:25] Many ways to describe signals

- Function e.g. $x(t) = A\cos(\omega_0 t + \phi)$
- Sequence of numbers, e.g. {1,2,3,2,1}
- Set of properties, e.g. even symmetric, causal
- Piecewise representation
- Generalized function, .e.g Dirac delta $\delta(t)$

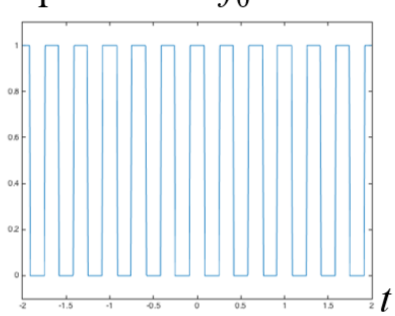
[11:30] Two-sided, one-sided, and finite length signals

Two-sided signals extend infinitely in positive and negative directions of time axis

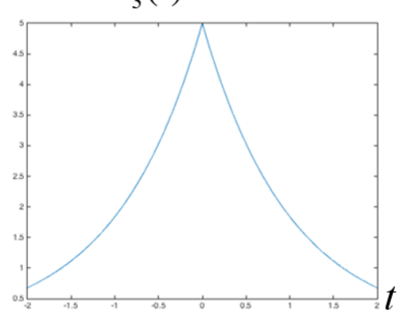
$x_1(t) = 10\cos(2\pi(1.5)t)$



Square wave $f_0 = 3$ Hz

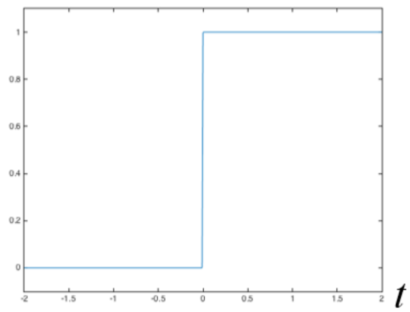


$x_3(t) = 5e^{-|t|}$

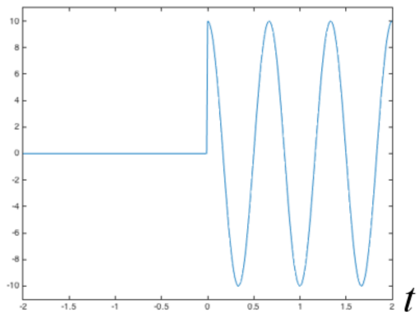


One-sided signals only extend in one direction of the time axis. Causal signals are equal to zero for $t < 0$.

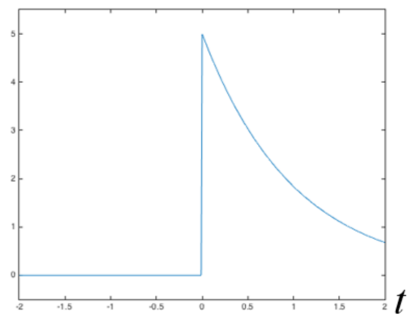
$x_1(t) = u(t)$



$x_2(t) = 10\cos(2\pi(1.5)t)u(t)$



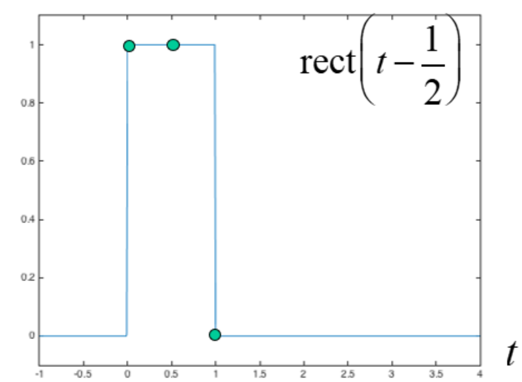
$x_3(t) = 5e^{-t}u(t)$

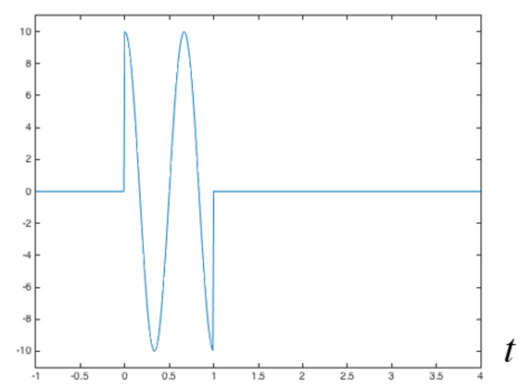


Finite length signals are equal to zero outside some time interval $t_1 \leq t \leq t_2$

$$\text{rect}(t) = \begin{cases} 1 & \text{if } -\frac{1}{2} \leq t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$x_2(t) = 10\ \cos(2\pi(1.5)t)\ \text{rect}\left(t - \frac{1}{2}\right)$$





[11:35] Unit impulse (Dirac Delta)

The Dirac Delta δ is an idealism for an instantaneous event. It is defined by two properties:

Unit area	Sifting property
$\int_{t=-\infty}^{\infty} \delta(t)dt = 1$	$\int_{t=-\infty}^{\infty} g(t)\delta(t)dt = g(0)$

The Dirac delta can also be expressed as a limit. Consider a rectangular pulse $P_\epsilon(t)$ with width 2ϵ and unit area:

$$P_\epsilon(t) = \frac{1}{2\epsilon}\operatorname{rect}\left(\frac{t}{2\epsilon}\right) = \begin{cases} \frac{1}{2\epsilon} & \epsilon \le t < \epsilon \\ 0 & \text{otherwise} \end{cases}$$

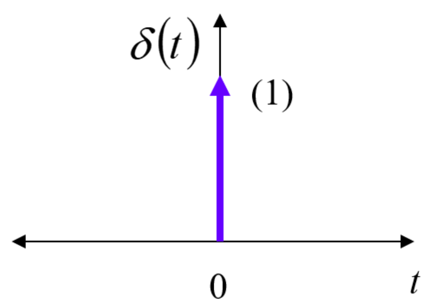
The Dirac Delta can be defined as the limit as the width of the pulse goes to zero, while keeping the area constant:

$$\delta(t) = \lim_{\epsilon \to 0} P_\epsilon(t)$$

$$\text{Area} = \lim_{\epsilon \to 0} \frac{2\epsilon}{2\epsilon} = 1$$

In practice, we can emulate the Dirac Delta using a finite width ϵ (e.g. if system operates on a time scale of seconds, use $\epsilon < 10^{-6}$ seconds) .

By convention, plot the dirac delta as a vertical area and denote its area using parentheses.



The direction of the arrow indicates the sign ($\delta(t)$ vs $-\delta(t)$)

[12:00] Properties of unit impulse

Generalized sifting, assuming $a > 0$

$$\int_{t=-a}^{a} \delta(t-T)dt = \begin{cases} 1 & \text{if} -a < T < a \\ 0 & \text{otherwise} \end{cases}$$

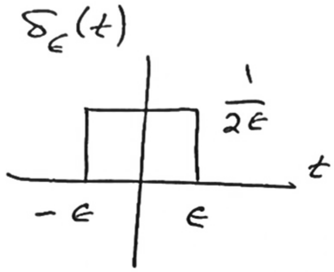
Consider the effect of integrating the product of a function $\phi(t)$ with $\delta(t)$:

$$\int_{t=-\infty}^{\infty} \phi(t)\delta(t-T)dt = \int_{t=-\infty}^{\infty} \phi(t+T)\delta(t)dt = \phi(T) \qquad \text{(sifting property)}$$

$$\int_{t=-\infty}^{\infty} x(\lambda)\delta(t-\lambda)dt = x(t) \qquad \text{(convolution with } \delta(t))$$

Slide 12-7 Dirac Delta 11/18/25

$$\text{rect}(t) = \begin{cases} 1 & \text{if } -\frac{1}{2} \le t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$



$$\delta_\epsilon(t) = \frac{1}{2\epsilon}\,\text{rect}\left(\frac{t}{2\epsilon}\right)$$

$$\text{rect}\left(\frac{t}{2\epsilon}\right) = \begin{cases} 1 & \text{if } -\frac{1}{2} \le \frac{t}{2\epsilon} < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

multiply by 2ϵ

$\epsilon > 0$

$$\delta(t) = \lim_{\epsilon \to 0} \delta_\epsilon(t)$$

$$\text{rect}\left(\frac{t}{2\epsilon}\right) = \begin{cases} 1 & \text{if } -\epsilon \le t < \epsilon \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Area} = \lim_{\epsilon \to 0} \frac{2\epsilon}{2\epsilon} = \lim_{\epsilon \to 0} \frac{2}{2} = 1$$

use L'Hôpital's Rule

[12:10] Continuous-time systems

Systems operate on signals to produce new signals or new signal representations.

$$x(t) \longrightarrow \boxed{T\{\bullet\}} \longrightarrow y(t)$$

$$y(t) = T\{\, x(t) \,\}$$

Additivity, homogeneity, and time-invariance definitions are the same as discrete-time:

Additivity	Homogeneity
Let $x_1(t)$ and $x_2(t)$ be arbitrary input signals to a system $\mathcal{S}$. The system satisfies additivity if $$\mathcal{S}\{x_1(t) + x_2(t)\} = \mathcal{S}\{x_1(t)\} + \mathcal{S}\{x_2(t)\}$$	Let $x(t)$ be an arbitrary input to a system $\mathcal{S}$. The system satisfies homogeneity if, for any constant a, $$\mathcal{S}\{ax(t)\} = a\mathcal{S}\{x(t)\}$$

Time-invariance	Linear time-invariant (LTI)
Let $x(t)$ be an arbitrary input to a system $\mathcal{S}$ and let $y(t) = \mathcal{S}\{x(t)\}$ be the corresponding output. The system $\mathcal{S}$ is time-invariant if, for any time shift τ, $$\mathcal{S}\{x(t-\tau)\} = y(t-\tau)$$	A system is linear time-invariant (LTI) if it satisfies the additivity, homogeneity, and time-invariance properties. A common way for a system to fail to violate these properties is if the system has has nonzero initial conditions.

[12:15] Initial conditions

Observe signals and systems starting at time $t = 0$.

Example: observe integrator for $t \geq 0$

$$y(t) = \int_{u=-\infty}^{t} x(u)\, du = \underbrace{\int_{u=-\infty}^{0} x(u)\, du}_{\text{initial condition}, C_0} + \int_{u=0}^{t} x(u)\, du$$

For $t \geq 0$, input signal $x(t)$ gives output

$$y(t) = C_0 + \int_0^t x(u)\, du$$

For linearity, the initial condition C_0 must be zero. We can see this from the all-zero input test—when $x(t) = 0$ for $t \geq 0$, the only way that $y(t) = 0$ for $t \geq 0$ is if $C_0 = 0$.

Homogeneous? Let the input be $a\, x(t)$ and the output be

$$y_{scaled}(t) = C_0 + \int_0^t (a\, x(u))\, du = C_0 + a \int_0^t x(u)\, du$$

When does $y_{scaled}(t) = a\, y(t)$ for all possible constants a? When $C_0 = a\, C_0$ or $C_0 = 0$.